SOLVING WAVE EQUATION IN A PLASMA $\sigma \neq 0$

We start with

$$-\nabla^2 \vec{E}_0 = \mu_0 \varepsilon_0 \omega^2 \vec{E}_0 - i\omega \mu_0 \vec{J}_0 - i\omega \mu_0 \sigma \vec{E}_0$$

Where $\vec{E} = \vec{E}_0 \exp(i\omega t)$, $\vec{E}_0$ = complex for phase

$$\vec{J}_{antenna} = \vec{J}_a \exp(i\omega t)$$

$$\vec{J}_{TOTAL} = \vec{J}_{antenna} + \vec{J}_{plasma}$$

$$\vec{J}_{plasma} = \sigma \vec{E}$$

Assume antenna is outside plasma and launches a wave into the plasma at $z=0$.

$$E_0(z) = E_0(z=0)\exp(-ikz)$$

In the plasma

$$-\nabla^2 E_0 \exp(-ikz) = (\mu_0 \varepsilon_0 \omega^2 - i\omega \mu_0 \sigma) E_0 \exp(-ikz)$$

$$-(ik)^2 E_0 = (\mu_0 \varepsilon_0 \omega^2 + i\omega \mu_0 \sigma) E_0$$

$$-k^2 = -\mu_0 \varepsilon_0 \omega^2 + i\omega \mu_0 \sigma$$

The speed of light in vacuum is $\mu_0 \varepsilon_0 = \frac{1}{c^2}$

The wave vector in vacuum is $\frac{\omega^2}{c^2} = \left(\frac{2\pi}{\lambda}\right)^2 = k_0^2$

$$-k^2 = \frac{-\omega^2}{c^2} + i\omega\mu_0\sigma$$

$$k^2 = \frac{\omega^2}{c^2}\left(1 - \frac{i\omega\mu_0\sigma c^2}{\omega^2}\right) = \frac{\omega^2}{c^2}\left(1 - \frac{i\mu_0\sigma c^2}{\omega^2}\right)$$

$$= k_0^2\left(1 - \frac{i\mu_0\sigma c^2}{\omega^2}\right)$$

Note that $\omega_{plasma}^2 = \frac{q^2 n_e}{\varepsilon_0 m_e}$, $\sigma = \frac{q^2 n_e}{m_e \nu_m}$

$$\omega_p^2 = \frac{\sigma\nu_m}{\varepsilon_0}, \quad \frac{\sigma}{\varepsilon_0} = \frac{\omega_p^2}{\nu_m}$$

So

$$k^2 = k_0^2\left(1 - i\frac{\omega_p^2}{\omega\nu_m}\right), \quad Q = \frac{\omega\nu_m}{\omega_p^2}$$

$$k = k_0\left(1 - \frac{i\omega_p^2}{\omega\nu_m}\right)^{1/2} = k_0\left(1 - \frac{i}{Q}\right)^{1/2} = k_r - ik_i$$

$$k_r = \frac{k_0}{\sqrt{2}}\left(\left(1 + \frac{1}{Q^2}\right)^{1/2} + 1\right)^{1/2}$$

$$k_i = \frac{k_0}{\sqrt{2}}\left(\left(1 + \frac{1}{Q^2}\right)^{1/2} - 1\right)^{1/2}$$

For a good conductor, $Q \ll 1$

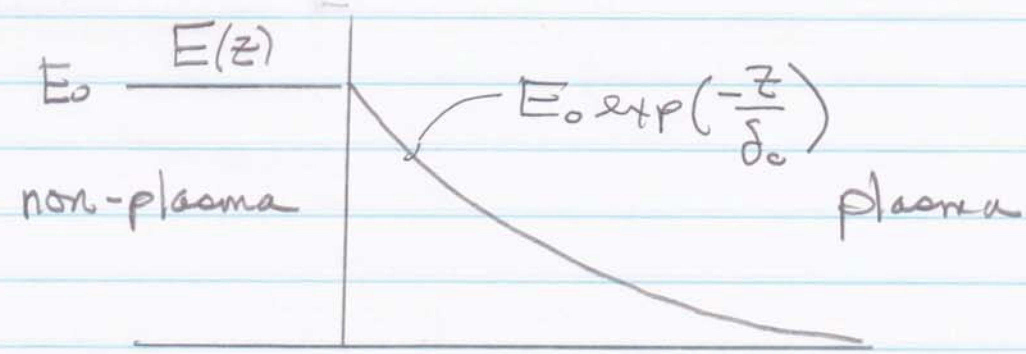
$$k_r \approx \frac{k_0}{\sqrt{2}} \frac{1}{Q^{1/2}} \qquad k_i = \frac{k_0}{\sqrt{2}} \frac{1}{Q^{1/2}}$$

$$k \approx \frac{k_0}{\sqrt{2}} \left(\frac{\omega_p^2}{\omega \nu_m}\right)^{1/2} (1-i) = \frac{\omega_p}{c} \left(\frac{\omega}{2\nu_m}\right)^{1/2} (1-i)$$

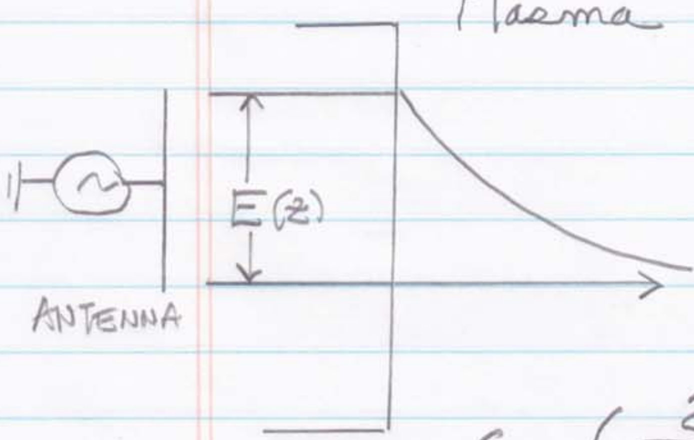
$$\approx \left(\frac{\sigma \nu_m}{c^2 \varepsilon_0} \frac{\omega}{2\nu_m}\right)^{1/2} (1-i) = \left(\frac{\sigma \omega}{2} \mu_0\right)^{1/2} (1-i)$$

$$k_i = \frac{1}{\delta_c}, \quad \delta_c = \left(\frac{2}{\sigma \omega \mu_0}\right)^{1/2}$$

$E(z) = E_0 \exp\left(\frac{-z}{\delta_c}\right)$, δ_c = collisional skin depth



Special Case



$$E(z) = E_o(z=0)\exp\left(\frac{-z}{\delta}\right)$$

δ_c = Collisional skin depth

$$\delta_c = \left(\frac{2}{\sigma\omega\mu_o}\right)^{1/2} = \left(\frac{2 m_e \nu_m}{q^2 n_e 2\pi\nu_{RF}\mu_o}\right)^{1/2}$$

σ = conductivity
μ_o = permeability
$\omega = 2\pi\nu_{RF}$

$$\sim \left(\frac{\nu_m}{\nu_{RF}}\frac{1}{n_e}\right)^{1/2}$$

- Highly conductive, high frequency $\Rightarrow$ low penetration depth but high power deposition

Example: Ar plasma, 10 mTorr, 13.56 MHz, $n_e = 10^{11}\,cm^{-3}$, $\nu_m = 10^7\,s^{-1}$

$\delta_c \approx 0.8\,cm$